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Constitutive relations for isotropic or kinematic hardening at finite elastic–plastic deformations

Li-Jun Shen *

Faculty of Engineering, Ningbo University, Ningbo, Zhejiang 315211, China

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Abstract

The rate-type constitutive relations of rate-independent metals with isotropic or kinematic hardening at finite elastic–plastic deformations were presented through a phenomenological approach. This approach includes the decomposition of finite deformation into elastic and plastic parts, which is different from both the elastic–plastic additive decomposition of deformation rate and Lee's elastic–plastic multiplicative decomposition of deformation gradient. The objectivity of the constitutive relations was dealt with in integrating the constitutive equations. A new objective derivative of back stress was proposed for kinematic hardening. In addition, the loading criteria were discussed. Finally, the stress for simple shear elastic–plastic deformation was worked out.

Keywords: Constitutive relation; Rate-independent; Finite elastic–plastic deformation; Isotropic and Kinematic hardening; Back stress; Objective derivative

1. Introduction

The phenomenological theories of finite elastic–plastic deformations have been thoroughly discussed by Nemat-Nasser (1992) and Naghdi (1990). In classical rate-independent plasticity, the finite elastic–plastic deformation was commonly supposed to be decomposed into elastic and plastic parts, which are prescribed through their respective constitutive laws. However, two issues remain unsettled. First, there are many disputes on the decomposition of deformation into elastic and plastic parts (see Naghdi, 1990). An error is believed to be introduced into the constitutive relations in the process of decomposing the total deformation into elastic and plastic parts (Metzger and Dubey, 1987). The existing rate-type constitutive relations

* Tel.: +86 574 87600302; fax: +86 574 87608358.
E-mail address: shenlijun@nbu.edu.cn

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are generally based on the additive decomposition of deformation rate into elastic and plastic parts. This decomposition of deformation rate is inconsistent with the multiplicative decomposition of deformation gradient into elastic and plastic parts. The choice of objective tensor rate was supposed to be the other issue in the rate-type constitutive relations. Researchers (for example, Dienes, 1979, 1986; Lee et al., 1983; Dafalias, 1983, 1985; Nemat-Nasser, 1983) have studied the constitutive relations for finite elastic–plastic deformations and proposed several objective tensor rates. Some of the classical objective tensor rates are Jaumann rate (material corotational rate), relative corotational rate (Green and Naghdi, 1965; Dienes, 1979) and Euler frame corotational rate (Sowerby and Chu, 1984). Szabó and Balla (1989) have compared and analyzed these objective stress rates. The rate-type constitutive relations using these objective rates for elastic–plastic deformations cannot be degenerated into the case of elastic deformation. Xiao et al. (1997) have taken the logarithmic stress rate as the objective stress rate to formulate a self-consistent hypoelastic model. The rate-type constitutive relations using the logarithmic stress rate, although applied to elastic deformations, cannot be proved to be applied to elastic–plastic deformations. The selection of objective tensor rate is somewhat arbitrary.

The two problems stated above will be dealt with in this paper. We will study the stress response to the sub-process of small deformation in the process of finite deformation and propose the decomposition of the finite deformation into elastic and plastic parts. This decomposition is different from both the multiplicative decomposition of deformation gradient and the additive decomposition of deformation rate. As a result, the constitutive relations for isotropic or kinematic hardening at finite elastic–plastic deformations will be presented through a phenomenological approach. The objective derivative of back stress will be studied in the case of kinematic hardening.

2. Constitutive relation for isotropic hardening at finite elastic–plastic deformations

2.1. The analysis of the existing constitutive relation

The deformation rate $\mathbf{D}$ is supposed to be decomposed as the sum of plastic (p) and elastic (e) parts, i.e.,

$$\mathbf{D} = \mathbf{D}_{\mathrm{p}} + \mathbf{D}_{\mathrm{e}} \tag{1}$$

which are respectively prescribed by the plastic flow rule (such as, the associated flow rule) and the generalized Hooke's law. Using the von Mises yield (or loading) criterion, we obtain the associated flow rule of plastic strain

$$\mathbf{D}_{\mathrm{p}} = \dot{\varphi}\mathbf{s} \tag{2}$$

where $\mathbf{s}$ is the deviatoric part of the Cauchy stress $\boldsymbol{\sigma}$, $\dot{\varphi}$ is the plastic multiplier. The generalized Prandtl–Reusss constitutive equation of isotropic hardening materials at finite deformations is supposed to be represented by

$$\mathbf{D} = \frac{\psi}{h}\frac{1}{\mathbf{s}:\mathbf{s}}(\overset{\circ}{\boldsymbol{\sigma}}:\mathbf{s})\mathbf{s} + \frac{1+\mu}{E}\overset{\circ}{\boldsymbol{\sigma}} - \frac{\mu}{E}(\mathrm{tr}\,\overset{\circ}{\boldsymbol{\sigma}})\mathbf{I} \tag{3}$$

where μ is Poisson's ratio, E Young's modulus, h a hardening factor, $\mathbf{I}$ the identity tensor and the notation tr() denotes tensor trace. $\psi = 1$ when the deforming body is loaded, $\psi = 0$ when the deforming body is unloaded. $\overset{\circ}{\boldsymbol{\sigma}}$ is an objective derivative (rate) of stress and is generally expressed in the form

$$\overset{\circ}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \boldsymbol{\Omega}\boldsymbol{\sigma} + \boldsymbol{\sigma}\boldsymbol{\Omega} \tag{4}$$

where $\boldsymbol{\Omega}$ is a rotational rate, a superposed dot denotes the material time derivative.

There are two problems in Eq. (3). First, it is not clear what kind of rotational rate should be used in Eq. (4). Many authors have proposed several rotational rates as plausible choice. Second, the additive decomposition of deformation rate is contradictory to Lee's decomposition of deformation gradient (see Lee, 1969; Naghdi, 1990).

According to the general theory of constitutive model (Truesdell, 1966), the constitutive relation of elastic isotropic materials is represented by

$$\boldsymbol{\sigma} = J_0\mathbf{I} + J_1\mathbf{V} + J_2\mathbf{V}^2 \tag{5}$$

where J_i $(i = 0, 1, 2)$ are scalar functions of three invariance of the left stretch tensor $\mathbf{V}$. Obviously, the Cauchy stress $\boldsymbol{\sigma}$ is coaxial with the left stretch tensor $\mathbf{V}$. The left stretch tensor is decomposed in the form $\mathbf{V} = \mathbf{R}_\mathrm{E}\mathbf{V}_\lambda\mathbf{R}_\mathrm{E}^\mathrm{T}$ where, in rectangular Cartesian coordinates, $\mathbf{R}_\mathrm{E}$ is a proper orthogonal matrix and $\mathbf{V}_\lambda$ is a diagonal matrix. The principal stress $\boldsymbol{\sigma}_\lambda$ depends on $\mathbf{V}_\lambda$. Hence, a constitutive relation of elastic deformation can be expressed in the form

$$\mathbf{D}_\mathrm{a} = \frac{1+\mu}{E}\overset{\circ}{\boldsymbol{\sigma}} - \frac{\mu}{E}(\mathrm{tr}\,\overset{\circ}{\boldsymbol{\sigma}})\mathbf{I} \tag{6}$$

where $\mathbf{D}_\mathrm{a} = \mathbf{R}_\mathrm{E}\overline{\ln\dot{\mathbf{V}}_\lambda}\mathbf{R}_\mathrm{E}^\mathrm{T}$, $\overset{\circ}{\boldsymbol{\sigma}} = \mathbf{R}_\mathrm{E}\overline{(\mathbf{R}_\mathrm{E}^\mathrm{T}\boldsymbol{\sigma}\mathbf{R}_\mathrm{E})}^{\boldsymbol{\cdot}}\mathbf{R}_\mathrm{E}^\mathrm{T}$, i.e., the rotation rate $\boldsymbol{\Omega}$ of the objective stress rate $\overset{\circ}{\boldsymbol{\sigma}}$ is $\dot{\mathbf{R}}_\mathrm{E}\mathbf{R}_\mathrm{E}^\mathrm{T}$.

Eq. (6) is equivalent to the hypoelastic model with the logarithmic rotational rate (Xiao et al., 1997, 1998; Bruhns et al., 1999). Hence, the constitutive equation (3) cannot be degenerated into the constitutive equation of finite elasticity if the rotational rate is the material or the relative or Euler frame rotational rate and not the logarithmic rotational rate. However, it has not yet been proved that Eq. (3) with the logarithmic rotational rate is applied to elastic–plastic deformations.

The material, the relative, Euler frame and the logarithmic rotational rates are determined only from the total deformation gradient. However, the rotational rate in Eq. (4) may not be entirely a kinematical quantity and may be related to the plastic part of the deformation gradient. The plastic deformation is dependent on the constitutive relation of elastic–plastic materials. We do not know how to deal with this cycle of calculation. Dafalias (1985, 1998) presented the plastic spin for anisotropic elastic–plastic deformation and proposed the constitutive spin (related to the plastic spin) used in the corotational rate of inner variables. However, he did not propose an explicit form of the constitutive relation of the plastic spin. It is difficult to choose objective rate even for such simple constitutive relations as Eq. (3). We will deal with the objectivity of the constitutive relation through a new approach in Section 2.2.

Next we analyze the decomposition of deformation into elastic and plastic parts. Researchers obtained the additive decomposition of deformation rate $\mathbf{D} = \mathbf{D}_\mathrm{e} + \mathbf{D}_\mathrm{ep}$ from the multiplicative decomposition of deformation gradient $\mathbf{F} = \mathbf{F}_\mathrm{e}\mathbf{F}_\mathrm{p}$ (Lee, 1969; Naghdi, 1990). They supposed that $\mathbf{D}_\mathrm{e}$ $(= \frac{1}{2}(\dot{\mathbf{F}}_\mathrm{e}\mathbf{F}_\mathrm{e}^{-1} + (\dot{\mathbf{F}}_\mathrm{e}\mathbf{F}_\mathrm{e}^{-1})^\mathrm{T}))$ is the elastic part of the deformation rate and $\mathbf{D}_\mathrm{ep}$ is the coupled elastic–plastic part of the deformation rate (Xiao et al., 2000) and is an approximate plastic deformation rate. However, the extent to which this approximation affects theoretical results remains unclear (Naghdi, 1990).

The deformation rate can be measured by taking intermediate (including the current and the initial) configuration as the reference state. Since the deformation gradient can be expressed in the form $\mathbf{F} = \mathbf{F}_1\mathbf{F}_0$ where $\mathbf{F}_1$ is measured by taking a certain intermediate configuration which may be the current configuration as the reference state, we obtain the velocity gradient $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \dot{\mathbf{F}}_1\mathbf{F}_1^{-1}$ and the deformation rate $\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^\mathrm{T})$. Hence, as in the case of infinitesimal deformation, we can obtain the decomposition of the deformation rate into elastic and plastic parts, $\mathbf{D} = \mathbf{D}_\mathrm{e} + \mathbf{D}_\mathrm{p}$, by taking the current configuration as the reference state. However, for the decomposition $\mathbf{F} = \mathbf{F}_\mathrm{e}$, $\mathbf{F}_\mathrm{p}$, $\mathbf{F}$ and $\mathbf{F}_\mathrm{p}$ are measured with reference to the initial configuration, while $\mathbf{F}_\mathrm{e}$ is measured with reference to the intermediate stress-free configuration. Therefore, the decomposition $\mathbf{D} = \mathbf{D}_\mathrm{e} + \mathbf{D}_\mathrm{p}$ is not derived from decomposition $\mathbf{F} = \mathbf{F}_\mathrm{e}\mathbf{F}_\mathrm{p}$. It is not surprising that the two decompositions contradict each other. If there exists the additive decomposition of the

Lagrangian strain into elastic and plastic parts, $\mathbf{E} = \mathbf{E}_e + \mathbf{E}_p$, $\mathbf{E}$, $\mathbf{E}_e$ and $\mathbf{E}_p$ should be measured with reference to the same configuration. So that we cannot obtain $\mathbf{E} = \mathbf{E}_e + \mathbf{E}_p$ from $\mathbf{F} = \mathbf{F}_e\mathbf{F}_p$, either.

If the decomposition $\mathbf{F} = \mathbf{F}_e\mathbf{F}_p$ is used, we cannot yet obtain the constitutive laws of $\mathbf{F}_e$ and $\mathbf{F}_p$ because the flow rule of plastic strain is expressed in the rate form. If the decomposition $\mathbf{D} = \mathbf{D}_e + \mathbf{D}_p$ is used, we do not know what kind of rotational rate should be used in the rate-type constitutive equations. The analysis of the decomposition of finite deformation into elastic and plastic parts will be continued in Section 2.2.

2.2. A new constitutive relation

Formulating now a new constitutive relation for isotropic hardening at finite elastic–plastic deformations, first we analyze the character of the elastic deformation of isotropic materials. A process of elastic deformation is subdivided into: $\mathbf{I} \to \mathbf{F}_1 \to \mathbf{F}_2 \to \mathbf{F}_3 \ldots$, where sub-indices $1, 2, 3, \ldots$ indicate respectively times $t_1, t_2, t_3, \ldots$ The deformation gradients are decomposed in the forms

$$\mathbf{F}_i = \mathbf{R}_{Ei}\mathbf{V}_{\lambda i}\mathbf{R}_{Li}^{\mathrm{T}} \quad (i = 1, 2, 3, \ldots) \tag{7}$$

where $\mathbf{V}_\lambda$ is a diagonal matrix whose components are the eigenvalues of left stretch tensor, both $\mathbf{R}_E$ and $\mathbf{R}_L$ are proper orthogonal matrices. Matrix denotes the rectangular Cartesian components of tensor in this paper. $\mathbf{V}_\lambda$, $\mathbf{R}_E$ and $\mathbf{R}_L$ in the two-dimension are respectively expressed in the forms

$$\mathbf{V}_{\lambda i} = \begin{bmatrix} a(i) & 0 \\ 0 & b(i) \end{bmatrix} \tag{8}$$

$$\mathbf{R}_{Ei} = \begin{bmatrix} \cos\theta(i) & -\sin\theta(i) \\ \sin\theta(i) & \cos\theta(i) \end{bmatrix} \tag{9}$$

$$\mathbf{R}_{Li} = \begin{bmatrix} \cos\beta(i) & -\sin\beta(i) \\ \sin\beta(i) & \cos\beta(i) \end{bmatrix} \tag{10}$$

where the independent variable i is time (t_i). Let $\theta(1) = 0$ with no basic loss of generality. Choosing an initial configuration, we have $\beta(1) = 0$. The deformation rate is decomposed in the form

$$\mathbf{D} = \mathbf{D}_a + \mathbf{D}_b \tag{11}$$

where

$$\mathbf{D}_a = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \dot{a}a^{-1} & 0 \\ 0 & \dot{b}b^{-1} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \mathbf{R}_E \overline{\ln \mathbf{V}_\lambda}^{\boldsymbol{\cdot}} \mathbf{R}_E^{\mathrm{T}}$$

$$\mathbf{D}_b = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0 & (ba^{-1} - ab^{-1})\dot{\beta}/2 \\ (ba^{-1} - ab^{-1})\dot{\beta}/2 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{12a, b}$$

Then, at time t_1, we have

$$\mathbf{D}_a = \begin{bmatrix} \dot{a}a^{-1} & 0 \\ 0 & \dot{b}b^{-1} \end{bmatrix}, \quad \mathbf{D}_b = \begin{bmatrix} 0 & (ba^{-1} - ab^{-1})\dot{\beta}/2 \\ (ba^{-1} - ab^{-1})\dot{\beta}/2 & 0 \end{bmatrix} \tag{13a, b}$$

Assume that the deformation whose gradient is a diagonal matrix generates a diagonal stress matrix. Then, the stress at time t_1 can be written as

$$\boldsymbol{\sigma}_{t_1} = \begin{bmatrix} \sigma_1(1) & 0 \\ 0 & \sigma_2(1) \end{bmatrix} \tag{14}$$

Obviously, the scalar product of the instantaneous stress $\boldsymbol{\sigma}_{t_1}$ and $\mathbf{D}_b$ of Eq. (13b) is equal to zero, that is, the power done by $\boldsymbol{\sigma}_{t_1}$ is independent of $\mathbf{D}_b$. Assume that the part $\mathbf{D}_b$ of the deformation rate does not affect the stress in the sub-process from t_1 to t_2. So that the variation of variable β and thus $\mathbf{R}_\mathrm{L}$ does not lead to the variation of the stress. The deformation rate sometimes implies the increment of small strain in this paper. Then, the stress at time t_2 is independent of $\mathbf{R}_\mathrm{L}$. According to the principle of objectivity of the constitutive model, the stress at t_2 can be written in the form

$$\boldsymbol{\sigma}_{t_2} = \begin{bmatrix} \cos\theta(2) & -\sin\theta(2) \\ \sin\theta(2) & \cos\theta(2) \end{bmatrix} \begin{bmatrix} \sigma_1(2) & 0 \\ 0 & \sigma_2(2) \end{bmatrix} \begin{bmatrix} \cos\theta(2) & \sin\theta(2) \\ -\sin\theta(2) & \cos\theta(2) \end{bmatrix} = \mathbf{R}_{\mathrm{E}2}\boldsymbol{\sigma}_{\lambda 2}\mathbf{R}_{\mathrm{E}2}^\mathrm{T} \tag{15}$$

Similarly, the deformation rate in the sub-process $\mathbf{F}_2 \to \mathbf{F}_3$ is also decomposed into two parts. The power done by the instantaneous stress is independent of one part $\mathbf{D}_\mathrm{b}$ of the deformation rate (from Eqs. (12b) and (15)). Thus, the stress at time t_3 can be written in the form

$$\sigma_{t_3} = \begin{bmatrix} \cos\theta(3) & -\sin\theta(3) \\ \sin\theta(3) & \cos\theta(3) \end{bmatrix} \begin{bmatrix} \sigma_1(3) & 0 \\ 0 & \sigma_2(3) \end{bmatrix} \begin{bmatrix} \cos\theta(3) & \sin\theta(3) \\ -\sin\theta(3) & \cos\theta(3) \end{bmatrix} = \mathbf{R}_{\mathrm{E}3}\sigma_{\lambda 3}\mathbf{R}_{\mathrm{E}3}^\mathrm{T} \tag{16}$$

It is seen from Eqs. (14)–(16) that the principal stress $\boldsymbol{\sigma}_\lambda$ depends on $\mathbf{V}_\lambda$ and the stress is coaxial with the left stretch tensor $\mathbf{V}$, which is consistent with the general theory of constitutive model (Eq. (5)). Hence, the assumption is rational that the deformation rate consists of two parts and the part $\mathbf{D}_\mathrm{b}$ of which the deformation power is independent does not affect the stress in the next sub-process.

The decomposition of Eq. (11) is easily generalized to the case of three dimensions. From Eq. (7), we obtain

$$\mathbf{D} = \mathbf{R}_\mathrm{E}\dot{\mathbf{V}}_\lambda\mathbf{V}_\lambda^{-1}\mathbf{R}_\mathrm{E}^\mathrm{T} + \tfrac{1}{2}\mathbf{R}_\mathrm{E}(\mathbf{V}_\lambda^{-1}\mathbf{R}_\mathrm{L}^\mathrm{T}\dot{\mathbf{R}}_\mathrm{L}\mathbf{V}_\lambda + \mathbf{V}_\lambda\dot{\mathbf{R}}_\mathrm{L}^\mathrm{T}\mathbf{R}_\mathrm{L}\mathbf{V}_\lambda^{-1})\mathbf{R}_\mathrm{E}^\mathrm{T} = \mathbf{D}_\mathrm{a} + \mathbf{D}_\mathrm{b} \tag{17}$$

Obviously, $\dot{\mathbf{V}}_\lambda\mathbf{V}_\lambda^{-1}$ is a diagonal matrix, $\frac{1}{2}(\mathbf{V}_\lambda^{-1}\mathbf{R}_\mathrm{L}^\mathrm{T}\dot{\mathbf{R}}_\mathrm{L}\mathbf{V}_\lambda + \mathbf{V}_\lambda\dot{\mathbf{R}}_\mathrm{L}^\mathrm{T}\mathbf{R}_\mathrm{L}\mathbf{V}_\lambda^{-1})$ is a symmetric matrix whose diagonal elements are all zero. Hence, the decomposition of Eq. (17) is the generalization of Eq. (11) in three dimensions.

The above analyses are not aimed at perfectly elastic deformations. In fact, both the hypoelastic model using the logarithmic stress rate and the constitutive equation (6) are consistent with the general theory of elasticity. The above assumption for elastic deformations can be generalized into elastic–plastic deformations.

In physical sense, a process of finite elastic–plastic deformation can be regarded as a series of sub-processes of small elastic–plastic deformation. Knowing the increments of stress in each sub-process and how to add these increments, we can formulate the constitutive relation. Consider a process of elastic–plastic deformation: $\mathbf{I} \to \mathbf{V}_{\lambda 1} \to \mathbf{F}_2 \to \mathbf{F}_3 \ldots$ It is at the elastic stage from time t_0 to time t_1, and reaches the elastic–plastic stage at time t_1. Then, the stress is a diagonal matrix $\boldsymbol{\sigma}_{\lambda 1}$ at time t_1. According to the associated flow rule, the plastic strain rate (deformation rate) must be a diagonal matrix in the sub-process $\mathbf{V}_{\lambda 1} \to \mathbf{F}_2 = \mathbf{R}_{\mathrm{E}2}\mathbf{V}_{\lambda 2}\mathbf{R}_{\mathrm{L}2}^\mathrm{T}$. From Eq. (15), only diagonal part of deformation rate affects the stress if $\mathbf{V}_{\lambda 1} \to \mathbf{F}_2$ is a process of elastic deformation. Hence, only diagonal deformation rate affects the stress in the sub-process of elastic–plastic deformation $\mathbf{V}_{\lambda 1} \to \mathbf{F}_2$. The assumption is valid even in the case of elastic–plastic deformation that the part $\mathbf{D}_\mathrm{b}$ of deformation rate (see Eq. (11) or (17)) of which the rate of the stress work is independent does not affect the stress in the next sub-process. Thus, the stress can be expressed in the form $\mathbf{R}_{\mathrm{E}2}\boldsymbol{\sigma}_{\lambda 2}\mathbf{R}_{\mathrm{E}2}^\mathrm{T}$ at t_2 as in the case of elastic deformation. From the generalized Hooke's law and the associated flow rule of plastic strain, we obtain the governing equation of principal stress $\boldsymbol{\sigma}_{\lambda 2}$

$$\dot{\overline{\ln\mathbf{V}_{\lambda 2}}} = \frac{\psi}{h}\frac{1}{\mathbf{s}_{\lambda 1} : \mathbf{s}_{\lambda 1}}(\dot{\boldsymbol{\sigma}}_{\lambda 2} : \mathbf{s}_{\lambda 1})\mathbf{s}_{\lambda 1} + \frac{1+\mu}{E}\dot{\boldsymbol{\sigma}}_{\lambda 2} - \frac{\mu}{E}(\mathrm{tr}\dot{\boldsymbol{\sigma}}_{\lambda 2})\mathbf{I} \tag{18}$$

where $\dot{\overline{\ln\mathbf{V}_{\lambda 2}}} = (\ln\mathbf{V}_{\lambda 2} - \ln\mathbf{V}_{\lambda 1})/(t_2 - t_1)$, $\dot{\boldsymbol{\sigma}}_{\lambda 2} = (\boldsymbol{\sigma}_{\lambda 2} - \boldsymbol{\sigma}_{\lambda 1})/(t_2 - t_1)$.

However, the principal direction of the stress need not be consistent with that of the left stretch tensor $\mathbf{V}$ in the case of elastic–plastic deformation. The stress need not be expressed in the form $\mathbf{R}_{\mathrm{E}3}\boldsymbol{\sigma}_{\lambda 3}\mathbf{R}_{\mathrm{E}3}^\mathrm{T}$ at time t_3.

The decomposition of deformation into elastic and plastic parts is supposed to be essential for the formulation of the constitutive relations for finite elastic–plastic deformations. Both the elastic–plastic decompositions $\mathbf{D} = \mathbf{D}_e + \mathbf{D}_p$ and $\mathbf{F} = \mathbf{F}_e\mathbf{F}_p$ are inappropriate for finite elastic–plastic deformations. In essence, the elastic Hooke's law is prescribed by taking the stress-free configuration as the reference state. The plastic flow rule is expressed in the rate form by taking the current configuration as the reference state. In the case of elastic deformation, the stress-free configuration is the initial configuration (perhaps a rigid body rotation is superposed). In the case of elastic–plastic deformation, the stress-free configuration may be an intermediate stress-free configuration.

The deformation gradient at time t_2 is decomposed into elastic (e) and plastic (p) parts in the form

$$\mathbf{R}_{E2}\mathbf{V}_{\lambda 2}\mathbf{R}_{L2}^{T} = (\mathbf{R}_{E2}\mathbf{V}_{\lambda 2e})(\mathbf{V}_{\lambda 2p}\mathbf{R}_{L2}^{T}) \tag{19}$$

where $\mathbf{R}_{E2}\mathbf{V}_{\lambda 2e}$ is the elastic deformation gradient generating the stress $\mathbf{R}_{E2}\boldsymbol{\sigma}_{\lambda 2}\mathbf{R}_{E2}^{T}$, $\mathbf{V}_{\lambda 2P}\mathbf{R}_{L2}^{T}$ is regarded as the measure of plastic (permanent) deformation. The deformation gradient at time t_3 is decomposed in the form

$$\mathbf{R}_{E3}\mathbf{V}_{\lambda 3}\mathbf{R}_{L3}^{T} = (\mathbf{R}_{E3b}\mathbf{V}_{\lambda 3b}\mathbf{R}_{L3b}^{T})(\mathbf{V}_{\lambda 2p}\mathbf{R}_{L2}^{T}) \tag{20}$$

Assume that the elastically or elastic–plastically deforming body intermediately unloaded and again loaded generates the same stress as this body continuously loaded does. Hence, we can take the intermediate stress-free configuration at t_2 as the reference state to study the sub-process from t_2 to t_3. From Eqs. (19) and (20), the deformation gradient should varies from $\mathbf{R}_{E2}\mathbf{V}_{\lambda 2e}$ to $\mathbf{R}_{E3b}\mathbf{V}_{\lambda 3b}\mathbf{R}_{L3b}^{T}$ from t_2 to t_3. An argument analogous to the deformation from t_1 to t_2 gives that the stress varies from $\mathbf{R}_{E2}\boldsymbol{\sigma}_{\lambda 2}\mathbf{R}_{E2}^{T}$ to $\mathbf{R}_{E3b}\boldsymbol{\sigma}_{\lambda 3}\mathbf{R}_{E3b}^{T}$ ($\mathbf{R}_{E3b}$ need not be equal to $\mathbf{R}_{E3}$). The governing equation of the principal stress is

$$\overline{\ln \mathbf{V}_{\lambda 3b}}^{\cdot} = \frac{\psi}{h}\frac{1}{\mathbf{s}_{\lambda 2} : \mathbf{s}_{\lambda 2}}(\dot{\boldsymbol{\sigma}}_{\lambda 3} : \mathbf{s}_{\lambda 2})\mathbf{s}_{\lambda 2} + \frac{1+\mu}{E}\dot{\boldsymbol{\sigma}}_{\lambda 3} - \frac{\mu}{E}(\mathrm{tr}\dot{\boldsymbol{\sigma}}_{\lambda 3})\mathbf{I} \tag{21}$$

Similarly, we have

$$\mathbf{R}_{E3b}\mathbf{V}_{\lambda 3b}\mathbf{R}_{L3b}^{T} = (\mathbf{R}_{E3b}\mathbf{V}_{\lambda 3be})(\mathbf{V}_{\lambda 3bp}\mathbf{R}_{L3b}^{T}) \tag{22}$$

where the deformation $\mathbf{R}_{E3b}\mathbf{V}_{\lambda 3be}$ is the elastic deformation generating the stress $\mathbf{R}_{E3b}\boldsymbol{\sigma}_{\lambda 3}\mathbf{R}_{E3b}^{T}$. A substitution of Eq. (22) into Eq. (20) yields

$$\mathbf{F} = (\mathbf{R}_{E3b}\mathbf{V}_{\lambda 3be})(\mathbf{V}_{\lambda 3bp}\mathbf{R}_{L3b}^{T})(\mathbf{V}_{\lambda 2p}\mathbf{R}_{L2}^{T}) = \mathbf{F}_{*e}\mathbf{F}_{*p} \tag{23}$$

where $(\mathbf{V}_{\lambda 3bp}\mathbf{R}_{L3b}^{T})(\mathbf{V}_{\lambda 2p}\mathbf{R}_{L2}^{T})$ is regarded as the measure of plastic deformation at time t_3. The deformation from t_3 to t_4 is dealt with in the same way as the above. The governing equations (18) and (21) etc. of principal stress and the expressions of the principal direction of stress represent the constitutive relation for elastic–plastic deformations.

Here, the decomposition of deformation is represented by $\mathbf{F} = \mathbf{F}_{*e}\mathbf{F}_{*p}$ where $\mathbf{F}_{*e}$ and $\mathbf{F}_{*p}$ respectively indicate "elastic" and "plastic" deformation. This decomposition and Lee's decomposition of deformation gradient are the same in the form but in essence different. In Lee's decomposition $\mathbf{F} = \mathbf{F}_e\mathbf{F}_p$, $\mathbf{F}_e$ and $\mathbf{F}_p$ are not supposed to be simultaneous, that is, the elastic and plastic deformation are the sub-processes in the whole process of deformation. In the decomposition $\mathbf{F} = \mathbf{F}_{*e}\mathbf{F}_{*p}$ proposed here, $\mathbf{F}_{*e}$ and $\mathbf{F}_{*p}$ are simultaneous in the whole process of deformation.

To degenerate the case of elastic–plastic deformation into the case of rigid-plastic deformation we consider a similar deformation: $\mathbf{I} \to \mathbf{F}_1 \to \mathbf{F}_2 \to \mathbf{F}_3$. The plastic deformation at t_1 is $\mathbf{V}_{\lambda 1}\mathbf{R}_{L1}^{T}$. The deformation gradient at t_2 is decomposed in the form $\mathbf{F}_2 = (\mathbf{R}_{E2b}\mathbf{V}_{\lambda 2b}\mathbf{R}_{L2b}^{T})(\mathbf{V}_{\lambda 1}\mathbf{R}_{L1}^{T})$. Hence, the stresses are $0 \to \mathbf{R}_{E1}\boldsymbol{\sigma}_{\lambda 1}\mathbf{R}_{E1}^{T} \to \mathbf{R}_{E2b}\boldsymbol{\sigma}_{\lambda 2}\mathbf{R}_{E2b}^{T}$, with

$$\overline{\ln \mathbf{V}_{\lambda 2\mathrm{b}}}^{\cdot} = \frac{1}{h}\frac{1}{\mathbf{s}_{\lambda 1} : \mathbf{s}_{\lambda 1}}(\dot{\boldsymbol{\sigma}}_{\lambda 2} : \mathbf{s}_{\lambda 1})\mathbf{s}_{\lambda 1} \tag{24}$$

We have $\mathbf{R}_{\mathrm{E2b}}\mathbf{V}_{\lambda 2\mathrm{b}}\mathbf{R}^{\mathrm{T}}_{\mathrm{L2b}} = \mathbf{F}_2\mathbf{F}_1^{-1}\mathbf{R}_{\mathrm{E1}}$. When $\mathbf{F}_2\mathbf{F}_1^{-1}$ trends to $\mathbf{I}$, we obtain

$$\mathbf{R}_{\mathrm{E2b}}\overline{\ln \mathbf{V}_{\lambda 2b}}^{\cdot}\mathbf{R}^{\mathrm{T}}_{\mathrm{E2b}} = \mathbf{D} \tag{25}$$

Thus Eq. (24) becomes

$$\mathbf{D} = \frac{1}{h}\frac{1}{\mathbf{s} : \mathbf{s}}(\dot{\boldsymbol{\sigma}} : \mathbf{s})\mathbf{s} \tag{26}$$

where $\dot{\boldsymbol{\sigma}} : \mathbf{s}$ is equal to $\overset{\circ}{\boldsymbol{\sigma}} : \mathbf{s}$. We mention in passing that in the case of rigid-plasticity the direction of the deformation rate $\mathbf{D}$ cannot suddenly change because the direction of the deformation rate must be consistent with that of the instantaneous deviatoric stress. In fact, there does not exist rigid-plastic deformation.

Next we analyze the transformations under superposed rigid body rotations of variables. When a rigid body rotation $\mathbf{Q}(t)$ ($\mathbf{Q}(0) = \mathbf{I}$) is superposed on the deformation, the deformation gradient transforms according to

$$\mathbf{F}^{+}(t) = \mathbf{Q}(t)\mathbf{F}(t) \tag{27}$$

From Eqs. (19), (20), (22), and (23), we obtain the transformations

$$\mathbf{F}^{+}_{*\mathrm{e}}(t) = \mathbf{Q}(t)\mathbf{F}_{*\mathrm{e}}(t), \quad \mathbf{F}^{+}_{*\mathrm{p}}(t) = \mathbf{F}_{*\mathrm{p}}(t) \tag{28a, b}$$

Thus, the stress transforms according to

$$\boldsymbol{\sigma}^{+}(t) = \mathbf{Q}(t)\boldsymbol{\sigma}(t)\mathbf{Q}^{\mathrm{T}}(t) \tag{29}$$

Therefore, the constitutive model proposed here obeys the principle of objectivity. The "plastic" deformation gradient is invariant under superposed rigid body rotations according to the above definition of the plastic deformation gradient.

If the deformation from $\mathbf{I}$ to $\mathbf{F} = \mathbf{R}_{\mathrm{E}}\mathbf{V}_{\lambda}\mathbf{R}^{\mathrm{T}}_{\mathrm{L}}$ is perfectly elastic deformation, $\mathbf{R}^{\mathrm{T}}_{\mathrm{L}}$ may be regarded as the plastic deformation. The plastic deformation $\mathbf{R}^{\mathrm{T}}_{\mathrm{L}}$ is invariant under superposed rigid body rotations.

From Eqs. (19), (20), (22), and (23), the transformations of the "elastic" and "plastic" deformation gradients can also be expressed in the forms

$$\mathbf{F}^{+}_{*\mathrm{e}}(t) = \mathbf{Q}(t)\mathbf{F}_{*\mathrm{e}}(t)\overline{\mathbf{Q}}^{\mathrm{T}}(*), \quad \mathbf{F}^{+}_{*\mathrm{p}}(t) = \overline{\mathbf{Q}}(*)\mathbf{F}_{*\mathrm{p}}(t) \tag{30a, b}$$

where $\overline{\mathbf{Q}}(*)$ is a proper orthogonal tensor and is independent of the rigid body rotation $\mathbf{Q}(t)$. It is found that neither Eqs. (30a,b) nor (28a,b) will lead to contradiction in the derivation of the constitutive relation.

Naghdi (1990) gave the transformations under superposed rigid body rotations of $\mathbf{F}_{\mathrm{e}}$ and $\mathbf{F}_{\mathrm{p}}$ in the decomposition $\mathbf{F} = \mathbf{F}_{\mathrm{e}}\mathbf{F}_{\mathrm{p}}$, i.e.,

$$\mathbf{F}^{+}_{\mathrm{e}}(t) = \mathbf{Q}(t)\mathbf{F}_{\mathrm{e}}(t)\overline{\mathbf{Q}}^{\mathrm{T}}(t), \quad \mathbf{F}^{+}_{\mathrm{p}}(t) = \overline{\mathbf{Q}}(t)\mathbf{F}_{\mathrm{p}}(t) \tag{31a, b}$$

where $\overline{\mathbf{Q}}(t)$ is a proper orthogonal tensor-valued function of time, different from $\mathbf{Q}(t)$.

If $\mathbf{D} = \mathbf{D}_{\mathrm{e}} + \mathbf{D}_{\mathrm{p}}$ is used, we have the transformations under superposed rigid body rotation $\mathbf{Q}(t)$ of $\mathbf{D}$, $\mathbf{D}_{\mathrm{e}}$ and $\mathbf{D}_{\mathrm{p}}$

$$\begin{aligned} &\mathbf{D}^{+}(t) = \mathbf{Q}(t)\mathbf{D}_{\mathrm{e}}(t)\mathbf{Q}^{\mathrm{T}}(t) + \mathbf{Q}(t)\mathbf{D}_{\mathrm{p}}(t)\mathbf{Q}^{\mathrm{T}}(t) \\ &\mathbf{D}^{+}_{\mathrm{e}}(t) = \mathbf{Q}(t)\mathbf{D}_{\mathrm{e}}(t)\mathbf{Q}^{\mathrm{T}}(t) \quad \text{and} \quad \mathbf{D}^{+}_{\mathrm{p}}(t) = \mathbf{Q}(t)\mathbf{D}_{\mathrm{p}}(t)\mathbf{Q}^{\mathrm{T}}(t) \end{aligned} \tag{32a, b, c}$$

From Eqs. (31b) and (32c), we show again that the decomposition $\mathbf{D} = \mathbf{D}_{\mathrm{e}} + \mathbf{D}_{\mathrm{p}}$ is not derived from the decomposition $\mathbf{F} = \mathbf{F}_{\mathrm{e}}\mathbf{F}_{\mathrm{p}}$. Similarly, we can show again that the additive decomposition of the Lagrangian strain into elastic and plastic parts ($\mathbf{E} = \mathbf{E}_{\mathrm{e}} + \mathbf{E}_{\mathrm{p}}$) is not derived from $\mathbf{F} = \mathbf{F}_{\mathrm{e}}\mathbf{F}_{\mathrm{p}}$.

We use Cauchy stress in the development above. The scalar product of Cauchy stress and deformation rate ($\boldsymbol{\sigma}:\mathbf{D}$) is the rate of stress work per unit volume at the instantaneous configuration, while $I_3\boldsymbol{\sigma}:\mathbf{D}$ is the rate of stress work per unit volume at the initial configuration where I_3 is the third invariant of the left stretch tensor. Some authors proposed that the weighted Cauchy stress $I_3\boldsymbol{\sigma}$, also called the Kirchhoff stress, replace the Cauchy stress in the constitutive equation for finite deformation. We can use the Kirchhoff stress instead of the Cauchy stress in the above development of the constitutive relation. However, the quantity I_3 should be measured with reference to intermediate stress-free configurations.

The constitutive relation proposed here sternly satisfies Hooke's law and the associated flow rule, and will not lead to contradiction in separating the total deformation into elastic and plastic parts.

3. Constitutive relation for kinematic hardening at elastic–plastic deformations

Take the Cartesian coordinate system making shear stress components vanish, the von Mises yield (or loading) surface with kinematic hardening is expressed as

$$f = \frac{1}{2}(\mathbf{s}-\boldsymbol{\alpha}):(\mathbf{s}-\boldsymbol{\alpha}) - \frac{1}{3}\sigma_y^2 = \frac{1}{2}(s_i - \alpha_{ij})(s_i - \alpha_{ij}) - \frac{1}{3}\sigma_y^2 = 0 \quad (i,j = 1,2,3) \tag{33}$$

where $\boldsymbol{\alpha}$ with components α_{ij} is the deviatoric back stress (abbreviated as back stress), (s_i) is a diagonal matrix whose components are eigenvalues of $\mathbf{s}$ and σ_y is the equivalent Cauchy stress at first yield. In general, $\alpha_{ij}(i \neq j)$ are not equal to zero.

In the past, the associated flow rule of plastic strain is represented by

$$\mathbf{D}_\mathrm{p} = \dot{\varphi}(\mathbf{s}-\boldsymbol{\alpha}) = \dot{\varphi}(s_i - \alpha_{ij}) \tag{34}$$

Several authors proposed the constitutive relation for kinematic hardening

$$\mathbf{D} = \frac{\psi}{h}\frac{1}{\bar{\mathbf{s}}:\bar{\mathbf{s}}}(\mathring{\boldsymbol{\sigma}}:\bar{\mathbf{s}})\bar{\mathbf{s}} + \frac{1+\mu}{E}\mathring{\boldsymbol{\sigma}} - \frac{\mu}{E}(\mathrm{tr}\,\mathring{\boldsymbol{\sigma}})\mathbf{I} \tag{35}$$

$$\mathring{\boldsymbol{\alpha}} = \frac{\mathring{\boldsymbol{\sigma}}:\bar{\mathbf{s}}}{\bar{\mathbf{s}}:\bar{\mathbf{s}}}\bar{\mathbf{s}} \tag{36}$$

where $\bar{\mathbf{s}} = \mathbf{s}-\boldsymbol{\alpha}$. Eq. (36) is referred to as the evolution equation of $\boldsymbol{\alpha}$. The objective derivatives of stress and back stress are supposed to have the same form. The above constitutive relation is based upon the decomposition of deformation rate. Hence, it is not applied to finite elastic–plastic deformations.

If the associated flow rule (34) was valid, the assumption for isotropic hardening could not be applied to kinematic hardening. In fact, the author's paper (2005) has shown that the associated flow rule should be

$$\mathbf{D}_\mathrm{p} = \dot{\varphi}(s_i - \alpha_{ii}) \tag{37}$$

where both (s_i) and (α_{ii}) are diagonal matrixes. It is noted that $(s_i - \alpha_{ii})$ is still an objective tensor. The plastic deformation rate is coaxial with $\bar{\mathbf{s}}_\mathrm{a} = (s_i - \alpha_{ii})$ and generally not coaxial with $\bar{\mathbf{s}} = (s_i - \alpha_{ij})$. Hence, $\mathbf{D}_\mathrm{p}$ is a diagonal matrix in the state of stress (σ_1, σ_2, σ_3).

As in the case of isotropic hardening, we still assume that the part of the deformation rate of which the rate of stress work is independent does not affect the stress. Hence, we can develop the constitutive relation for kinematic hardening in such a way as in the above section. When $\mathbf{s}_\lambda$ is replaced by $\bar{\mathbf{s}}_\mathrm{a} = (s_i - \alpha_{ii})$ the constitutive equations (18) and (21) for isotropic hardening becomes the constitutive equations for kinematic hardening. Consider a general elastic–plastic deformation: $\mathbf{I} \to \mathbf{V}_{\lambda 1} \to \mathbf{F}_2 \to \mathbf{F}_3 \ldots$, The deformation gradients are decomposed into elastic and plastic parts in such a form as Eqs. (19), (20), and (23). The stresses can be expressed in the forms $0 \to \boldsymbol{\sigma}_{\lambda 1} \to \mathbf{R}_{\mathrm{E}2}\boldsymbol{\sigma}_{\lambda 2}\mathbf{R}_{\mathrm{E}2}^\mathrm{T} \to \mathbf{R}_{\mathrm{E3b}}\boldsymbol{\sigma}_{\lambda 3}\mathbf{R}_{\mathrm{E3b}}^\mathrm{T}$. The governing equation of the principal stress in the sub-process from time t_2 to t_3 is, for example, expressed as

$$\overline{\ln \mathbf{V}_{\lambda 3\mathrm{b}}}^{\cdot} = \frac{\psi}{h}\frac{1}{\bar{\mathbf{s}}_\mathrm{a} : \bar{\mathbf{s}}_\mathrm{a}}(\dot{\boldsymbol{\sigma}}_{\lambda 3} : \bar{\mathbf{s}}_\mathrm{a})\bar{\mathbf{s}}_\mathrm{a} + \frac{1+\mu}{E}\dot{\boldsymbol{\sigma}}_{\lambda 3} - \frac{\mu}{E}(\mathrm{tr}\dot{\boldsymbol{\sigma}}_{\lambda 3})\mathbf{I} \tag{38}$$

where $\bar{\mathbf{s}}_\mathrm{a} = (s_i - \alpha_{ii})$ which is the quantity at time t_2. The above constitutive relation complies with the principle of objectivity.

The main objective of this section is to study the evolution equation of back stress. We analyze two objective derivatives of back stress and propose a new objective derivative of back stress. Consider the deformation: $\mathbf{I} \to \mathbf{R}_1\mathbf{U}_1 \to \mathbf{R}_2\mathbf{U}_2 \to \mathbf{R}_3\mathbf{U}_3 \ldots, \mathbf{U}_i$ $(i = 1, 2, 3, \ldots)$ are the right stretch tensors. Let $\boldsymbol{\alpha}_2$ and $\boldsymbol{\alpha}_3$ be respectively the back stresses at times t_2 and t_3.

(a) The relative corotational rate of back stress. Taking the initial configuration as the reference state, the deformation gradients at times t_2 and t_3 are respectively $\mathbf{R}_2\mathbf{U}_2$ and $\mathbf{R}_3\mathbf{U}_3$. Let us assume that $\boldsymbol{\alpha}_2$ is rotated by $\mathbf{R}_3\mathbf{R}_2^\mathrm{T}$ from t_2 to t_3, i.e.,

$$\boldsymbol{\alpha}_{23}^\mathrm{R} = (\mathbf{R}_3\mathbf{R}_2^\mathrm{T})\boldsymbol{\alpha}_2(\mathbf{R}_3\mathbf{R}_2^\mathrm{T})^\mathrm{T} \tag{39}$$

Then, the objective increment of back stress from t_2 to t_3 is defined by

$$\Delta^\mathrm{R}\boldsymbol{\alpha}_{23} = \boldsymbol{\alpha}_3 - (\mathbf{R}_3\mathbf{R}_2^\mathrm{T})^\mathrm{T}\boldsymbol{\alpha}_2(\mathbf{R}_3\mathbf{R}_2^\mathrm{T})^\mathrm{T} \tag{40}$$

and thus the corresponding objective derivative of back stress is defined by

$$\begin{aligned}\dot{\boldsymbol{\alpha}}^\mathrm{R} &= \lim_{t_3 - t_2 = \Delta t \to 0} \frac{\Delta^\mathrm{R}\boldsymbol{\alpha}_{23}}{\Delta t} \\ &= \lim_{\Delta t \to 0}\left\{(\mathbf{R}_3\mathbf{R}_2^\mathrm{T})\left(\frac{\boldsymbol{\alpha}_3 - \boldsymbol{\alpha}_2}{\Delta t}\right)(\mathbf{R}_3\mathbf{R}_2^\mathrm{T})^\mathrm{T} + \mathbf{R}_3\frac{(\mathbf{R}_3^\mathrm{T} - \mathbf{R}_2^\mathrm{T})}{\Delta t}\boldsymbol{\alpha}_3 + (\mathbf{R}_3\mathbf{R}_2^\mathrm{T})\boldsymbol{\alpha}_3\frac{(\mathbf{R}_3 - \mathbf{R}_2)}{\Delta t}\mathbf{R}_3^\mathrm{T}\right\} \\ &= \dot{\boldsymbol{\alpha}} + \boldsymbol{\alpha}\boldsymbol{\Omega}_\mathrm{R} - \boldsymbol{\Omega}_\mathrm{R}\boldsymbol{\alpha}\end{aligned} \tag{41}$$

where $\boldsymbol{\Omega}_\mathrm{R} = \dot{\mathbf{R}}\mathbf{R}^\mathrm{T}$, is the relative rotational rate. $\dot{\boldsymbol{\alpha}}^\mathrm{R}$ is the relative corotational rate. In a particular case, the back stress $\boldsymbol{\alpha}_0$ is not equal to zero at time t_0, the back stress is affected only by rotation after time t_0 (for instance, in the case of elastic deformation). Thus, the back stresses at times t_2 and t_3 are respectively

$$\boldsymbol{\alpha}_2 = \mathbf{R}_2\boldsymbol{\alpha}_0\mathbf{R}_2^\mathrm{T} \tag{42}$$

$$\boldsymbol{\alpha}_3 = \mathbf{R}_3\boldsymbol{\alpha}_0\mathbf{R}_3^\mathrm{T} = (\mathbf{R}_3\mathbf{R}_2^\mathrm{T})\boldsymbol{\alpha}_2(\mathbf{R}_3\mathbf{R}_2^\mathrm{T})^\mathrm{T} \tag{43}$$

Hence, the relative corotational rate implies that the rotation of back stress is calculated by taking the initial configuration as reference state.

(b) The material corotational rate of back stress (Jaumann rate). Taking current configuration as reference state, we can obtain the material corotational rate. Taking the configuration at time t_2 as reference state, we obtain the deformation gradients from t_2 to t_3, $\mathbf{I} \to \mathbf{R}_{23}\mathbf{U}_{23}$, where $\mathbf{R}_{23}\mathbf{U}_{23} = (\mathbf{R}_3\mathbf{U}_3)(\mathbf{R}_2\mathbf{U}_2)^{-1}$. Assume that $\boldsymbol{\alpha}_2$ is rotated by $\mathbf{R}_{23}$ from t_2 to t_3, i.e.,

$$\boldsymbol{\alpha}_{23}^\mathrm{J} = \mathbf{R}_{23}\boldsymbol{\alpha}_2\mathbf{R}_{23}^\mathrm{T} \tag{44}$$

Then, the objective increment of back stress from t_2 to t_3 is defined by

$$\Delta^\mathrm{J}\boldsymbol{\alpha}_{23} = \boldsymbol{\alpha}_3 - \mathbf{R}_{23}\boldsymbol{\alpha}_2\mathbf{R}_{23}^\mathrm{T} \tag{45}$$

and thus the corresponding objective derivative of back stress is defined by

$$\begin{aligned}\dot{\boldsymbol{\alpha}}^\mathrm{J} &= \lim_{t_3 - t_2 = \Delta t \to 0}\frac{\Delta^\mathrm{J}\boldsymbol{\alpha}_{23}}{\Delta t} = \lim_{\Delta t \to 0}\left\{(\mathbf{R}_{23})\left(\frac{\boldsymbol{\alpha}_3 - \boldsymbol{\alpha}_2}{\Delta t}\right)(\mathbf{R}_{23}^\mathrm{T}) + \frac{(\mathbf{I} - \mathbf{R}_{23})}{\Delta t}\boldsymbol{\alpha}_3 + \mathbf{R}_{23}\boldsymbol{\alpha}_3\frac{(\mathbf{I} - \mathbf{R}_{23}^\mathrm{T})}{\Delta t}\right\} \\ &= \dot{\boldsymbol{\alpha}} + \boldsymbol{\alpha}\mathbf{w} - \mathbf{w}\boldsymbol{\alpha}\end{aligned} \tag{46}$$

where $\mathbf{w}$ is the material rotational rate. $\dot{\boldsymbol{\alpha}}^{\mathrm{J}}$ is the material corotational rate. It should be noted that the material rotational rate is obtained by taking either initial or current configuration as reference state.

(c) A new objective derivative of back stress. Taking intermediate stress-free configuration at t_2 as reference state, the deformation gradients at t_2 and t_3 are respectively expressed in the forms

$$\mathbf{F}_2^{\mathrm{free}} = \mathbf{R}_{\mathrm{E2}}\mathbf{V}_{\lambda 2\mathrm{e}} \quad \text{and} \quad \mathbf{F}_3^{\mathrm{free}} = \mathbf{R}_{\mathrm{E3b}}\mathbf{V}_{\lambda 3\mathrm{b}}\mathbf{R}_{\mathrm{L3b}}^{\mathrm{T}} \tag{47a, b}$$

from Eqs. (19) and (20) used in the case of kinematic hardening. So we obtain the rotational tensor for $\boldsymbol{\alpha}_2$

$$\mathbf{P}_{23} = \mathbf{R}_{\mathrm{E3b}}\mathbf{R}_{\mathrm{L3b}}^{\mathrm{T}}\mathbf{R}_{\mathrm{E2}}^{\mathrm{T}}. \tag{48}$$

At time t_3, $\boldsymbol{\alpha}_2$ is rotated in the form

$$\boldsymbol{\alpha}_{23}^{\mathrm{p}} = \mathbf{P}_{23}\boldsymbol{\alpha}_2\mathbf{P}_{23}^{\mathrm{T}} \tag{49}$$

The objective increment of back stress from t_2 to t_3 is defined by

$$\Delta^{\mathrm{P}}\boldsymbol{\alpha}_{23} = \boldsymbol{\alpha}_3 - \mathbf{P}_{23}\boldsymbol{\alpha}_2\mathbf{P}_{23}^{\mathrm{T}} \tag{50}$$

and thus the corresponding objective derivative of back stress is defined by

$$\dot{\boldsymbol{\alpha}}^{\mathrm{p}} = \lim_{t_3 - t_2 = \Delta t \to 0} \frac{\Delta^{\mathrm{P}}\boldsymbol{\alpha}_{23}}{\Delta t} = \dot{\boldsymbol{\alpha}} + \boldsymbol{\alpha}\dot{\mathbf{P}}_{23} - \dot{\mathbf{P}}_{23}\boldsymbol{\alpha} \tag{51}$$

where $\dot{\mathbf{P}}_{23} = \lim_{t_3 - t_2 = \Delta t \to 0} \frac{(\mathbf{P}_{23} - \mathbf{I})}{\Delta t}$, is a skew-symmetric tensor. We have the transformation under superposed rigid body $\mathbf{Q}$ of $\dot{\mathbf{P}}_{23}$, $\dot{\mathbf{P}}_{23}^{+} = \mathbf{Q}\dot{\mathbf{P}}_{23}\mathbf{Q}^{\mathrm{T}} + \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}$. So that $\dot{\boldsymbol{\alpha}}^{\mathrm{p}}$ is the objective derivative of back stress. $\dot{\mathbf{P}}_{23}$ is not entirely a kinematical quantity and is related to plastic deformation gradient while both $\mathbf{w}$ and $\boldsymbol{\Omega}_{\mathrm{R}}$ are entirely kinematical quantities.

Suppose that the back stress $\boldsymbol{\alpha}_0$ is not equal to zero at time t_0. In the case of elastic deformation, Eqs. (47a,b) become

$$\mathbf{F}_2^{\mathrm{free}} = \mathbf{R}_{\mathrm{E2}}\mathbf{V}_{\lambda 2} \quad \text{and} \quad \mathbf{F}_3^{\mathrm{free}} = \mathbf{R}_{\mathrm{E3}}\mathbf{V}_{\lambda 3}\mathbf{R}_{\mathrm{L3}}^{\mathrm{T}}\mathbf{R}_{\mathrm{L2}} \tag{52a, b}$$

and thus the rotational tensor in Eq. (48) becomes

$$\mathbf{P}_{23} = \mathbf{R}_{\mathrm{E3}}\mathbf{R}_{\mathrm{L3}}^{\mathrm{T}}\mathbf{R}_{\mathrm{L2}}\mathbf{R}_{\mathrm{E2}}^{\mathrm{T}} = \mathbf{R}_3\mathbf{R}_2^{\mathrm{T}} \tag{53}$$

So Eq. (50) is the same as Eq. (40) and $\dot{\boldsymbol{\alpha}}^{\mathrm{p}}$ becomes the relative corotational rate $\dot{\boldsymbol{\alpha}}^{\mathrm{R}}$.

In the case of rigid-plastic deformation, Eqs. (47a,b) become

$$\mathbf{F}_2^{\mathrm{free}} = \mathbf{R}_{\mathrm{E2b}} \quad \text{and} \quad \mathbf{F}_3^{\mathrm{free}} = \mathbf{R}_{\mathrm{E3b}}\mathbf{V}_{\lambda 3\mathrm{b}}\mathbf{R}_{\mathrm{L3b}}^{\mathrm{T}} = \mathbf{F}_3\mathbf{F}_2^{-1}\mathbf{R}_{\mathrm{E2b}} \tag{54a, b}$$

and thus the rotational tensor of Eq. (48) becomes

$$\mathbf{P}_{23} = \mathbf{R}_{\mathrm{E3b}}\mathbf{R}_{\mathrm{L3b}}^{\mathrm{T}}\mathbf{R}_{\mathrm{E2b}}^{\mathrm{T}} = \mathbf{R}_{23} \tag{55}$$

where $\mathbf{R}_{23}$ is a proper orthogonal tensor in the polar decomposition $\mathbf{R}_{23}\mathbf{U}_{23} = \mathbf{F}_3\mathbf{F}_2^{-1}$. Hence, Eq. (50) becomes Eq. (45), $\dot{\boldsymbol{\alpha}}^{\mathrm{p}}$ becomes Jaumann rate $\dot{\boldsymbol{\alpha}}^{\mathrm{J}}$.

Using the Prager–Ziegler shifting model of the center of yield surface, assume that

$$\dot{\boldsymbol{\alpha}}^{\mathrm{P}} = \dot{m}\bar{\mathbf{s}} \tag{56}$$

where $\bar{\mathbf{s}} = \mathbf{s} - \boldsymbol{\alpha}$. According to the consistency condition of yield criterion (33), we obtain the evolution equation of back stress

$$\dot{\boldsymbol{\alpha}}^{\mathrm{P}} = \frac{\dot{\mathbf{s}}^{\mathrm{P}} : \bar{\mathbf{s}}}{\bar{\mathbf{s}} : \bar{\mathbf{s}}}\bar{\mathbf{s}} \tag{57}$$

In the same way as in the case of isotropic hardening, we obtain the constitutive relation for rigid-plastic deformation

$$\mathbf{D}_\lambda = \frac{1}{h}\frac{1}{\bar{\mathbf{s}}_a : \bar{\mathbf{s}}_a}(\dot{\mathbf{s}}_\lambda : \bar{\mathbf{s}}_a)\bar{\mathbf{s}}_a \quad \text{with } \boldsymbol{\sigma} = \mathbf{s} = \mathbf{R}_D\mathbf{s}_\lambda\mathbf{R}_D^T \tag{58}$$

where $\mathbf{D}_\lambda$ is a diagonal matrix containing the eigenvalues of deformation rate $\mathbf{D}$, $\mathbf{R}_D$ is a proper orthogonal matrix with $\mathbf{D} = \mathbf{R}_D\mathbf{D}_\lambda\mathbf{R}_D^T$. In the case of rigid-plastic deformation, the objective rates in Eq. (57) are the Jaumann rate. The Jaumann rate is supposed to lead to unreliable results, such as the oscillation of stress in the simple shear rigid-plastic deformation. However, this puzzle results from using the existing associated flow rule of plastic strain (expressed by Eq. (34)), but not from choosing Jaumann rate as the objective rate, as pointed out by the author's paper (2005).

4. The determination of the coefficient *h* for the constitutive relation

Uniaxial tension or compression can determine the scalar factor h in the constitutive relations for isotopic or kinematic hardening. In uniaxial tension of ductile metal bar, Cauchy stress on the cross-section of the bar is σ_1, the bar is l_0 long at the initial moment, l long at the current moment. From the constitutive relations for isotropic or kinematic hardening, we have

$$\overline{\ln(l/l_0)}^{\,\cdot} = \frac{2}{3h}\dot{\sigma}_1 + \frac{1}{E}\dot{\sigma}_1 \tag{59}$$

and

$$\frac{\mathrm{d}\ln(l/l_0)}{\mathrm{d}\sigma_1} = \frac{2}{3h} + \frac{1}{E} \tag{60}$$

We depict Cauchy stress σ_1—the logarithmic strain $\ln(l/l_0)$ curve for uniaxial tension from the examination. Let E be the slope of the cure at the elastic state, let E_t be the slope of the curve at the elastic–plastic phase, we obtain

$$\frac{1}{h} = \frac{3}{2}\left(\frac{1}{E_t} - \frac{1}{E}\right) \tag{61}$$

5. Loading criteria

In the case of isotropic hardening, the yield surface is $f = \frac{1}{2}\mathbf{s} : \mathbf{s} - \frac{1}{3}\sigma_y^2 = 0$. With σ_y fixed, the derivative of f with respect to time t is

$$\hat{f} = \dot{\mathbf{s}} : \mathbf{s} = \dot{\boldsymbol{\sigma}}_\lambda : \mathbf{s}_\lambda \tag{62}$$

From Eqs. (18) and (21), etc., we have

$$\overline{\ln \mathbf{V}_{\lambda 3b}}^{\,\cdot} : \mathbf{s}_\lambda = \left(\frac{3}{2E_t} - \frac{3}{2E} + \frac{1+\mu}{E}\right)(\dot{\boldsymbol{\sigma}}_\lambda : \mathbf{s}_\lambda) \tag{63}$$

For elastic–plastic uniaxial tension of work-hardening materials, we have $(\frac{3}{2E_t} - \frac{3}{2E} + \frac{1+\mu}{E}) > 0$, $(\dot{\boldsymbol{\sigma}}_\lambda : \mathbf{s}_\lambda) > 0$ and $\overline{\ln \mathbf{V}_{\lambda 3b}}^{\,\cdot} : \mathbf{s}_\lambda > 0$. For elastic–plastic uniaxial tension of work-softening materials, we have $(\frac{3}{2E_t} - \frac{3}{2E} + \frac{1+\mu}{E}) < 0$, $(\dot{\boldsymbol{\sigma}}_\lambda : \mathbf{s}_\lambda) < 0$ and $\overline{\ln \mathbf{V}_{\lambda 3b}}^{\,\cdot} : \mathbf{s}_\lambda > 0$. Hence the loading criteria are defined to be

(1) $\frac{1}{2}\mathbf{s}:\mathbf{s}-\frac{1}{3}\sigma_y^2=0$ and $(\dot{\boldsymbol{\sigma}}_\lambda:\mathbf{s}_\lambda)>0$ (only for work-hardening) or $\overline{\ln \dot{\mathbf{V}}_{\lambda 3\mathrm{b}}}:\mathbf{s}_\lambda>0$, loading;
(2) $\frac{1}{2}\mathbf{s}:\mathbf{s}-\frac{1}{3}\sigma_y^2=0$ and $\dot{\boldsymbol{\sigma}}_\lambda:\mathbf{s}_\lambda=0$ or $\overline{\ln \dot{\mathbf{V}}_{\lambda 3\mathrm{b}}}:\mathbf{s}_\lambda=0$, neutral loading;
(3) $\frac{1}{2}\mathbf{s}:\mathbf{s}-\frac{1}{3}\sigma_y^2=0$ and $\dot{\boldsymbol{\sigma}}_\lambda:\mathbf{s}_\lambda=0$ or $\overline{\ln \dot{\mathbf{V}}_{\lambda 3\mathrm{b}}}:\mathbf{s}_\lambda<0$, unloading from an elastic–plastic state;
(4) $\frac{1}{2}\mathbf{s}:\mathbf{s}-\frac{1}{3}\sigma_y^2<0$, elastic state.

Naghdi (1990) proposed that the loading criteria for work-softening materials should be constructed from yield surface in strain space.

In the case of kinematic hardening, the yield surface is $f=\frac{1}{2}(\mathbf{s}-\boldsymbol{\alpha}):(\mathbf{s}-\boldsymbol{\alpha})-\frac{1}{3}\sigma_y^2=0$. With $\boldsymbol{\alpha}$ and σ_y fixed, the derivative of f with respect to time t is

$$\hat{f}=\dot{\boldsymbol{\sigma}}_\lambda:\bar{\mathbf{s}}_\mathrm{a} \tag{64}$$

It should be noted that in differentiating f with respect to time t, the principal direction of stress should not be changed since that of back stress $\boldsymbol{\alpha}$ is not changed. $\hat{f}=\dot{\boldsymbol{\sigma}}_\lambda:\bar{\mathbf{s}}_\mathrm{a}=\dot{\mathbf{s}}_\lambda:\bar{\mathbf{s}}_\mathrm{a}$ is an objective scalar. So that $\hat{f}=\dot{\mathbf{s}}^\mathrm{P}:\bar{\mathbf{s}}=\overset{\circ}{\mathbf{s}}:\bar{\mathbf{s}}$, where $\overset{\circ}{\mathbf{s}}$ is any other kind of objective stress tensor. As in the case of isotropic hardening, the loading criteria are defined to be

(1) $\frac{1}{2}\bar{\mathbf{s}}:\bar{\mathbf{s}}-\frac{1}{3}\sigma_y^2=0$ and $\dot{\boldsymbol{\sigma}}_\lambda:\bar{\mathbf{s}}_\mathrm{a}>0$ (only for work-hardening) or $\overline{\ln \dot{\mathbf{V}}_{\lambda 3\mathrm{b}}}:\mathbf{s}_\mathrm{a}>0$, loading;
(2) $\frac{1}{2}\bar{\mathbf{s}}:\bar{\mathbf{s}}-\frac{1}{3}\sigma_y^2=0$ and $\dot{\boldsymbol{\sigma}}_\lambda:\bar{\mathbf{s}}_\mathrm{a}=0$ or $\overline{\ln \dot{\mathbf{V}}_{\lambda 3\mathrm{b}}}:\mathbf{s}_\mathrm{a}=0$, neutral loading;
(3) $\frac{1}{2}\bar{\mathbf{s}}:\bar{\mathbf{s}}-\frac{1}{3}\sigma_y^2=0$ and $\dot{\boldsymbol{\sigma}}_\lambda:\bar{\mathbf{s}}_\mathrm{a}<0$ or $\overline{\ln \dot{\mathbf{V}}_{\lambda 3\mathrm{b}}}:\mathbf{s}_\mathrm{a}<0$, unloading from an elastic–plastic state;
(4) $\frac{1}{2}\mathbf{s}:\mathbf{s}-\frac{1}{3}\sigma_y^2<0$, elastic state.

In the above loading criteria, using $\mathbf{V}_{\lambda 3b}$ does not imply that the loading criteria are appropriate only for the process $\mathbf{F}_2\rightarrow\mathbf{F}_3$. The loading criteria are appropriate for any sub-processes by using the quantities corresponding to the sub-processes.

The evolution equation of the back stress is on the basis of the consistency condition of yield criteria. Hence, if $\frac{1}{2}\bar{\mathbf{s}}:\bar{\mathbf{s}}-\frac{1}{3}\sigma_y^2=0$ and loading or neutral loading, Eq. (57) is valid, otherwise, $\overset{\circ}{\boldsymbol{\alpha}}=0$. It is noted that the deviatoric back stress may be rotated in elastic deformations though $\overset{\circ}{\boldsymbol{\alpha}}$ is equal to zero.

6. Simple shear deformation

The simple shear deformation is acted as an example in order to compare the theory in this paper with other theories. The motion in simple shear is

$$x_1=X_1+kX_2,\quad x_2=X_2,\quad x_3=X_3 \tag{65}$$

where x_i and X_i $(i=1,2,3)$ are respectively rectangular Cartesian coordinates of the current and initial configuration. We obtain the deformation gradient $\mathbf{F}$, the deformation rate $\mathbf{D}$, the materials rotational rate $\mathbf{w}$, the relative rotational rate $\boldsymbol{\Omega}_\mathrm{R}=\dot{\mathbf{R}}\mathbf{R}^\mathrm{T}$ and Euler frame rotational rate $\boldsymbol{\Omega}_\mathrm{E}=\dot{\mathbf{R}}_\mathrm{E}\mathbf{R}_\mathrm{E}^\mathrm{T}$ (see the relevant references, for example, Shen, submitted for publication). The logarithmic rotational rate (Bruhns et al., 1999) is

$$\boldsymbol{\Omega}_\mathrm{Log}=\frac{\dot{k}}{4}\left(\frac{4}{1+\dot{k}}+\frac{k}{\sqrt{1+k^2}sh^{-1}k/2}\right)\begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix} \tag{66}$$

We calculate the stress response to the simple shear of the elastic–plastic materials exhibiting isotropic hardening by respectively using the model presented in the paper and the other models with two

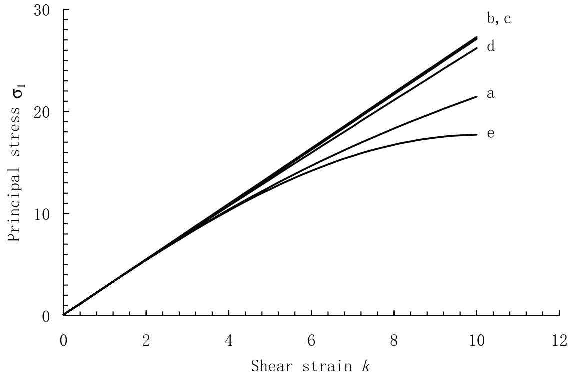

Fig. 1. Principal stress response during simple shear for elastic–plastic materials exhibiting isotropic hardening (the equivalent stress at initial yield is equal to 0.16 GPa, $\mu = 0.3$, $E = 39$ GPa, $E_t = 7.96$ GPa). Curve (a) is based on the new theory. Curves (b–e) are based on Eq. (3) where the rotation rate is respectively taken as $\mathbf{\Omega}_{\mathrm{E}}$, $\mathbf{\Omega}_{\mathrm{R}}$, $\mathbf{\Omega}_{\mathrm{Log}}$ and $\mathbf{w}$.

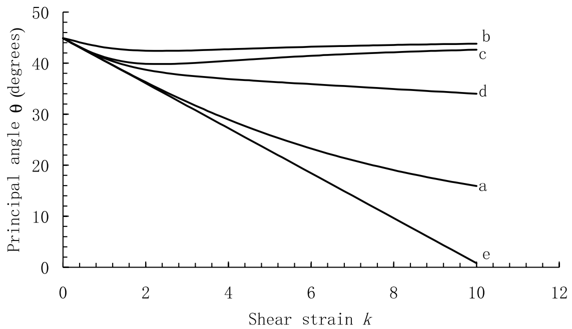

Fig. 2. Principal angle response during simple shear for elastic–plastic materials exhibiting isotropic hardening (the equivalent stress at initial yield is equal to 0.16 GPa, $\mu = 0.3$, $E = 39$ GPa, $E_t = 7.96$ GPa). Curve (a) is based on the new theory. Curves (b–e) are based on Eq. (3) where the rotation rate is respectively taken as $\mathbf{\Omega}_{\mathrm{E}}$, $\mathbf{\Omega}_{\mathrm{R}}$, $\mathbf{\Omega}_{\mathrm{Log}}$ and $\mathbf{w}$.

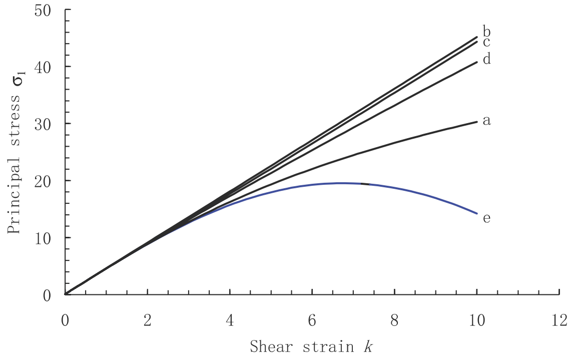

Fig. 3. Principal angle response during simple shear for elastic–plastic materials exhibiting isotropic hardening (the equivalent stress at initial yield is equal to 0.16 GPa, $\mu = 0.3$, $E = 39$ GPa, $E_t = 13$ GPa). Curve (a) is based on the new theory. Curves (b–e) are based on Eq. (3) where the rotation rate is respectively taken as $\mathbf{\Omega}_{\mathrm{E}}$, $\mathbf{\Omega}_{\mathrm{R}}$, $\mathbf{\Omega}_{\mathrm{Log}}$ and $\mathbf{w}$.

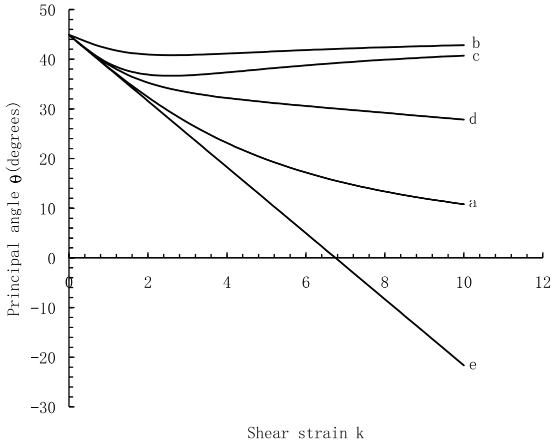

Fig. 4. Principal angle response during simple shear for elastic–plastic materials exhibiting isotropic hardening (the equivalent stress at initial yield is equal to 0.16 GPa, $\mu = 0.3$, $E = 39$ GPa, $E_t = 13$ GPa). Curve (a) is based on the new theory. Curves (b–e) are based on Eq. (3) where the rotation rate is respectively taken as $\mathbf{\Omega}_{\mathrm{E}}$, $\mathbf{\Omega}_{\mathrm{R}}$, $\mathbf{\Omega}_{\mathrm{Log}}$ and $\mathbf{w}$.

ratios of Young's modules E to the slope E_t. The principal stresses σ_1 and the angles θ are depicted in Figs. 1–4.

It is shown in the figures that the principal angle may not monotonically vary with shear strain if the constitutive equation (3) associated respectively with $\mathbf{w}$, $\mathbf{\Omega}_{\mathrm{R}}$ and $\mathbf{\Omega}_{\mathrm{E}}$ is used. Both the principal stress and the principal angle monotonically vary with shear strain if the new model or the constitutive equation (3) with $\mathbf{\Omega}_{\mathrm{Log}}$ is used. The larger shear strain k, the greater the difference between the new theory and other theories.

In addition, the new constitutive relations of the isotropic hardening and the kinematic hardening materials lead to the same result for the simple shear.

7. Conclusions

1. The constitutive relations with isotropic and kinematic hardening at finite elastic–plastic deformations are presented through a new approach. These constitutive relations are represented by a series of increment type equations, instead of one rate-type equation that is a combination of the associated flow rule and the hypoelastic model. The constitutive relations sternly satisfies the Hooke's law and the associated flow rule, which do not lead to contradiction in the separation of the total deformation into elastic and plastic parts.
2. A new objective rate (derivative) of back stress is presented for kinematic hardening. The form of this objective rate is generally related to plastic deformation gradient. The objective rate becomes Jaumann rate in the case of rigid-plastic deformation.

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